

Hamilton's equations of motion of a vortex filament in the rotating Bose-Einstein condensate and their "soliton" solutions

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The equation of motion of a quantized vortex filament in a trapped Bose-Einstein condensate [A. A. Svidzinsky and A. L. Fetter, Phys. Rev. A **62**, 063617 (2000)] has been generalized to the case of an arbitrary anharmonic anisotropic rotating trap and presented in a variational form. For condensate density profiles of the form $\rho = f(x^2 + y^2 + \text{Re} \Psi(x + iy))$ in the presence of the plane of symmetry $y = 0$, the solutions $x(z)$ describing stationary vortices of U and S types coming to the surface and solitary waves have been found in quadratures. Analogous three-dimensional configurations of the vortex filament uniformly moving along the z axis have also been found in strictly cylindrical geometry. The dependence of solutions on the form of the function $f(q)$ has been analyzed.

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Introduction. The dynamics of quantized vortex filaments is an important part of Bose condensed gas physics (see, e.g., reviews [1, 2] and numerous references therein). At low temperatures, a Bose-Einstein condensate is described well by the Gross-Pitaevskii equation

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi + V(\mathbf{r}_{\text{lab}}, t)\psi + g|\psi|^2\psi. \quad (1)$$

At rather fast rotation of the trap potential $V(\mathbf{r}_{\text{lab}}, t)$, the vortex-free state of the quantum gas of atoms with the mass m is energetically unfavorable and the vortex with the circulation $\Gamma = 2\pi\hbar/m$ penetrates inside the condensate. The vortex filament with its own velocity field exists against the background of the nonuniform unperturbed (mass) density $\rho_0 = m|\psi_0|^2$ and the unperturbed velocity \mathbf{v}_0 , which are time-independent in the coordinate system rotating about the z axis:

$$\mathbf{v}_0 = \Omega(\nabla\varphi - [\mathbf{e}_z \times \mathbf{r}]), \quad \nabla \cdot (\rho_0 \mathbf{v}_0) = 0. \quad (2)$$

We assume that the rotation frequency Ω is much lower than the characteristic transverse frequency of the trap ω_\perp , which provides the stability of the unperturbed flow [3, 4]. Then, the density field $\rho_0(\mathbf{r})$ in the Thomas-Fermi approximation and ignoring the terms of order $(\Omega/\omega_\perp)^2$ is determined from the equation

$$V(\mathbf{r}) + (g/m)\rho_0(\mathbf{r}) \approx \mu = \text{const}, \quad (3)$$

where $V(\mathbf{r})$ is the trap potential in the rotating system and μ is the rather large chemical potential of the gas satisfying the condition $\mu \gg \hbar\omega_\perp$. We note that the characteristic transverse size of the condensate is $R_0 \sim (\mu/m\omega_\perp^2)^{1/2}$, while the characteristic thickness of the vortex core is $\xi \sim \hbar/(m\mu)^{1/2}$. Their ratio is large, $R_0/\xi \sim \mu/\hbar\omega_\perp \gg 1$, which makes it possible to use the

so-called local induction approximation in the dynamics of the vortex filament. The condition $\rho_0(\mathbf{r}) = 0$ determines the (usually closed) Thomas-Fermi surface Σ , which can be considered as an effective boundary of the condensate. In practice, the topology of the surface Σ is most often spherical or toroidal (see, e.g., [5, 6]) but also can be more complicated as a function of the trap potential: the condensate may have several through holes and/or internal cavities.

Since the vortex filament is transported by the total velocity field, its equation of motion in the local induction approximation has the form

$$\mathbf{R}_t = \frac{\Gamma\Lambda}{4\pi} \{ \kappa \mathbf{b} + [(\nabla\rho_0(\mathbf{R})/\rho_0(\mathbf{R})) \times \mathbf{t}] \} + \mathbf{v}_0(\mathbf{R}), \quad (4)$$

where $\Lambda = \log(R_0/\xi) \approx \log(\mu/\hbar\omega_\perp) \approx \text{const} \gg 1$ is the large logarithm, κ is the local curvature of the filament, \mathbf{t} is the tangent unit vector, and \mathbf{b} is the binormal unit vector. For the harmonic trap, this equation was derived in [7] using crosslinked asymptotic expansions and then it was used in the linearized form in several works [8, 9]. It is significant that the Hamiltonian structure of Eq. (4) was not discussed earlier except for the case $\Omega = 0$ (see [10], where the local induction equation against the static inhomogeneous background was derived from the noncanonical Hamiltonian formalism). However, it is known that the stationary solutions of Eq. (4) in the case of the quadratic potential are extremals of a certain functional [11].

This work is devoted to the formulation of a variational principle for Eq. (4) in the general case and the determination of analytical nonlinear solutions for some (rather broad) class of density profiles uniform over z . Although real Bose-Einstein condensates are limited in the axial direction, the analytical solutions obtained may be approximately correct for rather long condensates.

Variational formulation. Let the position of the vortex filament in space be given by a vector function $\mathbf{R}(\beta, t)$ with an arbitrary longitudinal parameter β . We note that the existence of a vector potential for unper-

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turbed current density follows from Eq. (2):

$$\rho_0(\nabla\varphi - [\mathbf{e}_z \times \mathbf{r}]) = \text{curl } \mathbf{A}. \quad (5)$$

The vector potential $\mathbf{A}(\mathbf{r})$ satisfies the equation

$$\text{curl} \frac{1}{\rho_0(\mathbf{r})} \text{curl } \mathbf{A} = -2\mathbf{e}_z. \quad (6)$$

It is easy to check that Eq. (4) can be rewritten as follows (see details in [10] for the case $\Omega = 0$):

$$\Gamma [\mathbf{R}_\beta \times \mathbf{R}_t] \rho_0(\mathbf{R}) = \delta\mathcal{H}/\delta\mathbf{R}, \quad (7)$$

where the Hamiltonian functional is

$$\mathcal{H} = \frac{\Gamma^2}{4\pi} \Lambda \int \rho_0(\mathbf{R}) |\mathbf{R}_\beta| d\beta + \Gamma \Omega \int (\mathbf{A}(\mathbf{R}) \cdot \mathbf{R}_\beta) d\beta. \quad (8)$$

Under the boundary condition $\mathbf{A}|_\Sigma = 0$, this functional is approximately (owing to the relation $R_0/\xi \gg 1$) the part of the total kinetic energy of the flow that depends on the position of the vortex filament. It is important that Eq. (7) follows from the variational principle with the Lagrangian

$$\mathcal{L} = \Gamma \int (\mathbf{D}(\mathbf{R}) \cdot [\mathbf{R}_\beta \times \mathbf{R}_t]) d\beta - \mathcal{H}, \quad (9)$$

and the vector function $\mathbf{D}(\mathbf{R})$ satisfies the equation

$$\nabla \cdot \mathbf{D}(\mathbf{R}) = \rho_0(\mathbf{R}). \quad (10)$$

Thus, the required variational formulation of the dynamics of a single vortex filament is found in terms of the vector potential \mathbf{A} . The generalization for several filaments is formally simple but is technically difficult, since it is necessary to take into account the nonlocal interaction between filaments; to this end, it is necessary to know the solution of the equation

$$\text{curl} \frac{1}{\rho_0(\mathbf{r})} \text{curl } \mathbf{A}_\Gamma = \mathbf{\Omega}_\Gamma, \quad (11)$$

where $\mathbf{\Omega}_\Gamma$ is the singular vorticity field created by filaments and \mathbf{A}_Γ is the vector potential of the corresponding current density.

If necessary, using results of [12], a variational principle in the dynamics of the vortex filament can also be formulated in the case of a completely time-dependent background density profile. This means that the trap potential not only rotates but also is deformed. We do not discuss this problem here.

Case of the global stream function. For the further more particular consideration, it is necessary to have the explicit solution of Eq. (6). We consider the case where the unperturbed velocity field is two-dimensional (independent of the coordinate z) and solenoidal. This means that there exists the stream function of the unperturbed velocity common for all z values:

$$\Theta(x, y) = \frac{\Omega}{2} [x^2 + y^2 + \theta(x, y)], \quad \theta_{xx} + \theta_{yy} = 0, \quad (12)$$

where the harmonic function $\theta(x, y)$ determines the axial asymmetry of the flow. The density field in this case has the form

$$\rho_0(\mathbf{r}) = f(x^2 + y^2 + \theta(x, y); z), \quad (13)$$

where $f(q; z)$ is a rather arbitrary function of two variables. We introduce the function $Q(z)$ determined by the condition $f(Q; z) = 0$ and the function

$$F(q; z) = \int_q^{Q(z)} f(u; z) du, \quad (14)$$

which is positive inside the condensate and is zero at its boundary. Then, the vector potential is

$$\mathbf{A}(\mathbf{r}) = -\frac{\mathbf{e}_z}{2} F(x^2 + y^2 + \theta(x, y); z). \quad (15)$$

If the condensate has no holes, the harmonic function $\theta(x, y)$ can be represented in the form of the sum

$$\theta(x, y) = \text{Re} \sum_{n \geq 2} c_n (x + iy)^n \quad (16)$$

with arbitrary complex coefficients c_n . If the surface Σ is not topologically equivalent to a sphere, the diversity of analytical functions of the complex variable $\Psi(x + iy)$ makes it possible to consider rather exotic forms of condensates, including those with the quantized circulations of the unperturbed velocity $l_j \Gamma$ around each hole.

It should be noted that the quadratic anisotropic trap corresponds to the choice $\theta = \epsilon(x^2 - y^2)$ and

$$f_{quadr} \propto 1 - (1 + \epsilon)x^2 - (1 - \epsilon)y^2 - z^2/z_{max}^2.$$

Here and below, all lengths are given in units of R_0 and all times are measured in units of $2\pi R_0^2/\Gamma\Lambda \gg 1/\omega_\perp$.

We choose the coordinate z as the longitudinal parameter β . Then, $\mathbf{R} = (x(z, t), y(z, t), z)$ and the dimensionless Hamiltonian is written in the form

$$H = \frac{1}{2} \int f(x^2 + y^2 + \theta(x, y); z) \sqrt{1 + x_z^2 + y_z^2} dz - \frac{\tilde{\Omega}}{2} \int F(x^2 + y^2 + \theta(x, y); z) dz, \quad (17)$$

where $\tilde{\Omega} = 2\pi R_0^2 \Omega/\Gamma\Lambda$. The corresponding noncanonical Hamiltons equations of motion have the structure

$$x_t = \frac{1}{f} \frac{\delta H}{\delta y}, \quad -y_t = \frac{1}{f} \frac{\delta H}{\delta x}. \quad (18)$$

Analytical solutions. Stationary (in the rotating coordinate system) configurations of the vortex filament are extremals of the functional H . In a number of cases, they can be calculated analytically, e.g., when f is independent of z and all coefficients c_n are real. The vortex filament lies in the plane of symmetry $y = 0$, and the minimizing functional has the form

$$2\hat{H} = \int f(\alpha(x)) \sqrt{z_\beta^2 + x_\beta^2} d\beta - \tilde{\Omega} \int F(\alpha(x)) z_\beta d\beta, \quad (19)$$

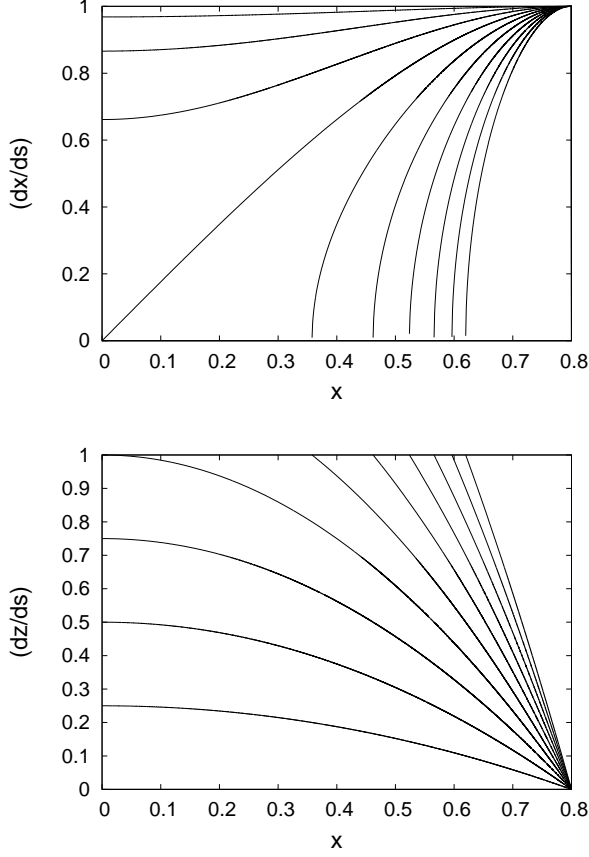


FIG. 1: Examples of phase curves for U and S vortices at different $\tilde{\Omega}$ values. At $\tilde{\Omega} < 2$, S vortices occur (reaching the value $x = 0$); At $\tilde{\Omega} > 2$, U vortices exist. At $\tilde{\Omega} = 2$, half the limiting soliton without self-crossing is obtained.

where $\alpha(x) = x^2 + \sum_{n \geq 2} c_n x^n$. The corresponding Euler-Lagrange equations are integrated:

$$f \frac{dz}{ds} - \tilde{\Omega} F = E = \text{const}, \quad (20)$$

$$\left(\frac{dx}{ds} \right)^2 = 1 - \left(\frac{E + \tilde{\Omega} F}{f} \right)^2, \quad (21)$$

where ds is the arc length element. In the case of zero integration constant E , the so-called U and S vortices coming on the surface are obtained (see, e.g., [13-15]). If $E = f(0) - \tilde{\Omega} F(0)$, the solutions are soliton-like dependences $x(z)$; in some cases, these can be curves with self-crossing. Self-crossing solutions certainly violate the condition of applicability of the local induction approximation and, therefore, they are not physical. Corresponding examples are given in Figs. 1 and 2 for the simplest choice $q = (1 + \epsilon)x^2 + (1 - \epsilon)y^2$, $\epsilon = 9/16$, $f(q) = 1 - q$.

Cylindrical geometry. The solution of the problem is the most advanced in the presence of cylindrical

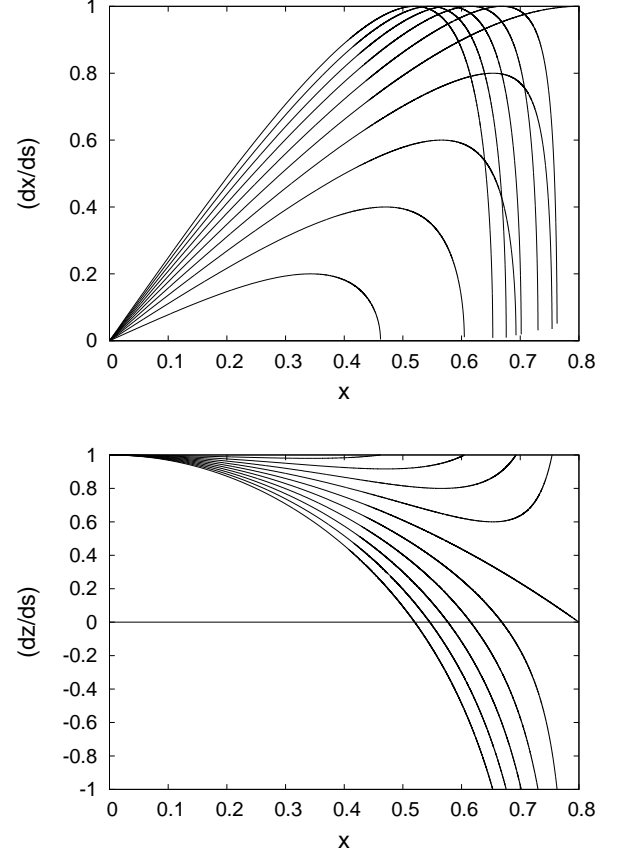


FIG. 2: Examples of phase curves for solitons at different $\tilde{\Omega}$ values. At $1 < \tilde{\Omega} < 2$, solutions are “good”; at $\tilde{\Omega} > 2$, curves with self-crossing are obtained.

symmetry. The density profile is static in the laboratory coordinate system and it is not necessary to transfer to the rotating system for the formulation of the variational principle. It is convenient to introduce the complex unknown variable $w(z, t) = r(z, t)e^{i\phi(z, t)} = x(z, t) + iy(z, t)$. Now, the vector function $\mathbf{D}(\mathbf{R})$ has the explicit form

$$\mathbf{D}(\mathbf{R}) = \frac{G(r^2)}{2r} (\cos \phi, \sin \phi, 0), \quad (22)$$

where

$$G(q) = \int_0^q f(u) du = F(0) - F(q). \quad (23)$$

is a nonnegative function. The Lagrangian of the filament is determined by the expression

$$L_* = \int \frac{i}{2} G(|w|^2) \left[\frac{w_t}{w} - \frac{w_t^*}{w^*} \right] dz - \int f(|w|^2) \sqrt{1 + |w_z|^2} dz, \quad (24)$$

and its equation of motion has the form

$$iw_t = \frac{f'(|w|^2)}{f(|w|^2)} \left[\frac{2w - w_z^2 w^* + |w_z|^2 w}{2\sqrt{1 + |w_z|^2}} \right] - \frac{\partial}{\partial z} \left(\frac{w_z}{2\sqrt{1 + |w_z|^2}} \right). \quad (25)$$

Integrals of motion of this equation (in addition to the Hamiltonian $H_* = \int f(|w|^2) \sqrt{1 + |w_z|^2} dz$) are given by the formulas

$$N = \int G(|w|^2) dz = \text{const}, \quad (26)$$

$$P = - \int \frac{i}{2} G(|w|^2) \left[\frac{w_z}{w} - \frac{w_z^*}{w^*} \right] dz = \text{const}. \quad (27)$$

Solutions of the form $w = r(z - vt) \exp[i\phi(z - vt) + i\tilde{\Omega}t]$ give the extremum to the functional

$$\tilde{H}_* = H_* + \tilde{\Omega}N - vP = \int (f ds + \tilde{\Omega}G dz - vG d\phi), \quad (28)$$

where $ds = \sqrt{dz^2 + dr^2 + r^2 d\phi^2}$. The formulas below follow from the requirement $\delta\tilde{H}_* = 0$

$$\frac{d\phi}{ds} = \frac{vG + M}{r^2 f}, \quad \frac{dz}{ds} = \frac{C - \tilde{\Omega}G}{f}, \quad (29)$$

with constant M and C . Then, we obtain

$$\left(\frac{dr}{ds} \right)^2 = \frac{f^2(r^2) - [vG(r^2) + M]^2/r^2 - [C - \tilde{\Omega}G(r^2)]^2}{f^2(r^2)}. \quad (30)$$

It is noteworthy that the curve at $v \neq 0$ or at $M \neq 0$ is necessarily three-dimensional. For three-dimensional ‘‘solitons,’’ the integration constants are $M = 0$ and $C = f(0)$. For three-dimensional U and S vortices, $M = -vG(r_{max}^2)$ and $C = \tilde{\Omega}G(r_{max}^2)$.

The small-angle approximation $|w_z|^2 \ll 1$ deserves special attention, since it allows qualitative analysis of the dependence of the dynamics on the density profile. In this case, the Hamiltonian and equation of motion are given by the formulas

$$\mathcal{H}_{(2)} = \int f(|w|^2) \left(1 + \frac{|w_z|^2}{2} \right) dz, \quad (31)$$

$$iw_t = -\frac{1}{2}w_{zz} + \frac{f'(|w|^2)}{f(|w|^2)} \left[w - \frac{w_z^2 w^*}{2} \right]. \quad (32)$$

At small $|w|^2$ values, we have $f'(|w|^2)/f(|w|^2) \approx c_0 + c_1|w|^2$. If $c_1 < 0$ and $c_1 > 0$, Eq. (32) in the long-wavelength limit is similar to the focusing and defocusing nonlinear Schrödinger equations, respectively. We also note that the Gaussian density profile $f(r^2) = \exp(-r^2)$ gives a quite compact nonlinear Schrödinger-type equation but with another cubic nonlinearity:

$$iw_t = -\frac{1}{2}w_{zz} - w + \frac{w_z^2 w^*}{2}. \quad (33)$$

Conclusions. Thus, the variational formulation of the dynamics of the vortex filament in the rotating Bose-Einstein condensate made it possible to find analytically some stationary solutions. In particular, in addition to the known vortices of the S and U types, the soliton-like configurations, which apparently have not yet been discussed, have been obtained. Noncanonical Hamiltonian equations of motion derived in this work can be used further for the numerical simulation of the dynamics of the filament. The interaction of solitons can possibly lead to the phenomenon of anomalous waves (rogue waves) and be accompanied by the approach of the vortex filament on the surface with its subsequent rupture.

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